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Should Controls Be Eliminated While Solving Optimal Control Problems via Direct Methods?

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Direct methods of solving optimal control problems include techniques based on control discretization, where the control function of time is parameterized, and collocation, where both the control and state functions of time are parameterized. A recently introduced direct approach of solving optimal control problems via differential inclusions parameterizes only the state, and constrains the state rates to lie in a feasible hodograph space. In this method, the controls, which are just artifacts used to parameterize the feasible hodograph space, are completely eliminated from the optimization process. Explicit and implicit schemes of control elimination are discussed. Comparison of the differential inclusions method is made to collocation in terms of number of parameters, number of constraints, CPU time required for solution, and ease of calculation of analytical gradients. A minimum time-to-climb problem for an F-15 aircraft is used as an example for comparison. For a special class of optimal control problems with linearly appearing bounded controls, it is observed that the differential inclusion scheme is better in terms of number of parameters and constraints. Increased robustness of the differential inclusion methodology over collocation for the Goddard problem with singular control as part of the optimal solutions is also observed.

Background

THE most precise approach to solve optimal control problems is the variational[1] approach based on Pontryagin's minimum principle.[2] This is an indirect approach as it involves solving the necessary conditions of optimality associated with the infinite dimensional optimal control problem rather than optimizing the cost of a finite dimensional discretization of the original problem directly. This method requires advanced analytical skills and to generate numerical solutions of the resulting two-point boundary-value problem is highly nontrivial. The controls are eliminated in the indirect method using the minimum principle. Thus, the optimal control is, in general, a nonlinear function of the state and costate variables. The most important application of the indirect method is the generation of benchmark solutions. Usually, good convergence is achieved only with excellent initial guesses for the nonintuitive costates. Additionally, the switching structure has to be guessed correctly in advance.

For rapid trajectory prototyping, the safest approaches are the direct methods.[3] These methods rely on a finite dimensional discretization of the optimal control problem to a nonlinear programming problem. Even though these methods do not enjoy the high precision and resolution of indirect methods, their convergence robustness makes them the method of choice of most practical applications. Moreover, these methods do not require the advanced mathematical skills necessary to pose and solve the variational problem.

Introduction

The most obvious approach to solving optimal control problems using direct methods is to discretize only the control functions of time. This method is implemented in the program to optimize simulated trajectories (POST) software[4] developed by Martin Marietta. Even though the number of parameters is kept reasonably small, this method exhibits only moderate convergence robustness. Typically, the states in the later part of the trajectory are very sensitive to changes in the control variables in earlier parts of the trajectory.

Excellent convergence is achieved using collocation based methods[5] in which both states and controls are discretized. Instead of precise integration of the state equations, the differential equations are enforced only at discrete points. The remaining parameters in the discretized problem are then used to directly optimize the cost function. This approach is implemented, for instance, in the optimal trajectories by implicit simulation (OTIS) code originally developed by Boeing and currently being enhanced by Boeing and McDonnell Douglas.

A recently introduced approach[6] is based on a representation of the dynamical system in terms of differential inclusions.[7] Rather than using the customary differential equations representation for the dynamic system, this method exploits the concepts of hodograph space[8] and attainable sets.[9] Only the states are discretized and the state rates at discrete points are constrained to lie within the feasible hodograph space. The method is completely devoid of controls, and initial guesses are required only for the states. Especially for singular optimal control problems, the absence of the fast-moving controls seems to have a positive effect on the convergence robustness. A user-friendly version of the current software package Trajectory Optimization via Differential Inclusions (TODI) with the differential-inclusion approach as the centerpiece is currently under development by the authors.

The next section formulates a minimum time-to-climb problem[10] of an F-15 aircraft, and the sections following compare the collocation-based approach to the differential inclusion approach. Both explicit and implicit control elimination schemes are utilized and contrasted for the differential-inclusion approach.

F-15 Climb-to-Dash Problem

The problem of steering an F-15 aircraft using bounded throttle setting $\eta(t)$ and bounded vertical load factor $n(t)$ from prescribed initial conditions to the farmost point to the right of the level flight envelope (dash-point) in minimum time is used to compare and contrast the solution approaches discussed in the preceding sections. Explicitly the minimum time-to-climb problem may be stated in Mayer form as follows.

Minimize t_f subject to the state equations of motion

$$\dot{h} = v \sin\gamma \tag{1}$$

$$\dot{E} = (\eta T - D)(v/mg) \tag{2}$$

$$\dot{\gamma} = (g/v)(n - \cos\gamma) \tag{3}$$

the control constraints

$$0 \le \eta \le 1 \tag{4}$$

$$-n_{\max} \le n \le +n_{\max} \tag{5}$$

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and the initial and terminal conditions

$$\begin{aligned} h(0) &= 5 & h(t_f) &= 12119.3 \\ E(0) &= 2668 & E(t_f) &= 38029.2 \\ \gamma(0) &= 0 & \gamma(t_f) &= 0 \end{aligned} \tag{6}$$

where the altitude h (in meters), the specific energy E (in meters) replacing velocity v, and the flight-path angle γ (in radians) are the state variables. The mass m (in kilograms) of the aircraft and the gravitational acceleration g (in meters per second square) are assumed to be constant, namely,

$$m = 16818 \quad \text{and} \quad g = 9.80665$$

The velocity v is defined by $v = \sqrt{[2g(E-h)]}$.

The initial states are picked rather arbitrarily and represent the state of the aircraft shortly after takeoff. The final states represent level flight conditions at the dashpoint. The associated state variables are found by solving the following maximization problem:

$$\underset{h,E,\gamma,n,\eta}{\text{maximize}}\ v$$

subject to level-flight conditions

$$\gamma = 0, \qquad \dot{\gamma} = 0, \qquad \dot{E} = 0$$

The atmospheric density and speed of sound are functions of altitude. Aerodynamic lift is proportional to load factor, and the drag D is dependent on velocity and altitude and has quadratic dependence on load factor. The maximum thrust T is also a function of altitude and velocity. Detailed atmospheric, aerodynamic, and propulsive models used for this study can be found in Refs. 10 and 11.

Collocation Approach

The precision of the problem discretization associated with the direct approach is controlled through a user-chosen integer N, through which $N + 1$ (for simplicity, equidistantly placed) nodes

$$t_i = t_f \cdot (i/N), \qquad i = 0, \ldots, N$$

are defined along the time axis. Then the values of the states $[h_i, E_i, \gamma_i]^T$ defined at the nodes t_i, $i = 0, \ldots, N$, the piecewise constant controls $[\eta_i, n_i]^T$, $i = 0, \ldots, N-1$ defined along each nodal interval $[t_i, t_{i+1}]$, $i = 0, \ldots, N-1$, and the final time t_f become parameters of a nonlinear programming problem. The cost to be minimized is t_f subject to the boundary constraints

$$\begin{aligned} h_0 &= 5 & h_N &= 12119.3 \\ E_0 &= 2668 & E_N &= 38029.2 \\ \gamma_0 &= 0 & \gamma_N &= 0 \end{aligned} \tag{7}$$

and differential and control constraints

$$\dot{h}_i - \bar{v}_i \sin\bar{\gamma}_i = 0 \tag{8}$$

$$\dot{E}_i - (\eta_i \bar{T}_i - \bar{D}_i)(\bar{v}_i/mg) = 0 \tag{9}$$

$$\dot{\gamma}_i - (n_i - \cos\bar{\gamma}_i)(g/\bar{v}_i) = 0 \tag{10}$$

$$0 \le \eta_i \le 1 \tag{11}$$

$$-n_{\max} \le n_i \le n_{\max} \tag{12}$$

where for $i = 0, \ldots, N-1$,

$$\begin{aligned} \bar{h}_i &= \frac{h_{i+1} + h_i}{2} & \dot{h}_i &= \frac{h_{i+1} - h_i}{\Delta t} \\ \bar{E}_i &= \frac{E_{i+1} + E_i}{2} & \dot{E}_i &= \frac{E_{i+1} - E_i}{\Delta t} \\ \bar{\gamma}_i &= \frac{\gamma_{i+1} + \gamma_i}{2} & \dot{\gamma}_i &= \frac{\gamma_{i+1} - \gamma_i}{\Delta t} \end{aligned} \tag{13}$$

and $\Delta t = t_f/N$. The expression $\bar{D}_i$ in Eq. (9) denotes the drag evaluated at the states $\bar{E}_i$ and $\bar{h}_i$ and at the load factor n_i corresponding to that nodal interval. Similarly, the thrust $\bar{T}_i$ and velocity $\bar{v}_i$ are evaluated at the states $\bar{E}_i$, $\bar{h}_i$.

Conditions (7) represent the initial and final boundary conditions (6), and constraints (8–12) enforce the differential equations and control constraints at discrete points. It can be expected that if Δt is small, then the Euler approximation (13) is adequate to obtain a first-cut, near-optimal solution. Costate initial guesses required to develop variational solutions with precise integration can be easily generated, for example, with the methods described in Ref. 12.

For the present example, the number of parameters for the nonlinear programming problem associated with collocation assuming piecewise constant controls is $3(N+1) + 2N + 1$ and the number of constraint equations is given by $5N + 6$. Hence, for $N = 10$, the number of parameters is 54, and the number of constraints is 56. The resulting nonlinear programming problem is solved using NPSOL.[13] The results obtained, convergence behavior, computer run-time statistics, etc., are discussed in a later section.

Differential Inclusion Approach

The approach is based upon a representation of the dynamical system in terms of differential inclusions. By exploiting the concepts of admissible state rates and attainable sets to describe the evolution of the states, the controls are completely eliminated from the model representation and the optimization process. Only the states are discretized and the state rates at discrete points (approximated by finite differences) are constrained to lie within the hodograph space as shown in Fig. 1.

Besides reducing the dimensionality of the discretized problem, compared to the state-of-the-art collocation methods, this approach also alleviates the search for initial guesses from where standard gradient search methods are able to converge. Absence of controls in the optimization problem for the differential inclusion approach has the added advantage that no initial guess for the control variables needs to be provided. Note, however, that conversion of the optimal control problem into a differential inclusion format is not always easy. This represents a clear disadvantage of this method.

The resulting nonlinear programming problem for the minimum time-to-climb problem via differential inclusions is as follows: values of the states $[h_i, E_i, \gamma_i]^T$ defined at the nodes t_i, $i = 0, \ldots, N$ and the final time t_f become parameters. The cost to be minimized is t_f subject to the constraints (7), and the hodograph constraints

$$\dot{h}_i - \bar{v}_i \sin\bar{\gamma}_i = 0 \tag{14}$$

$$0 \le \frac{\dot{E}_i(mg/\bar{v}_i) + \bar{D}_i}{\bar{T}_i} \le 1 \tag{15}$$

$$-n_{\max} \le [\dot{\gamma}_i(\bar{v}_i/g) + \cos\bar{\gamma}_i] \le +n_{\max} \tag{16}$$

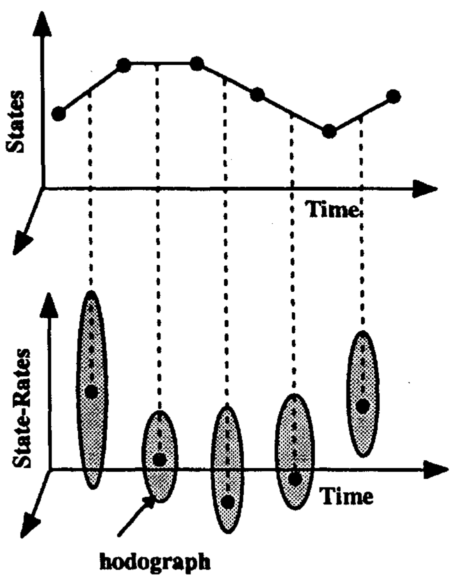

Fig. 1 Schematic of the differential inclusion algorithm.

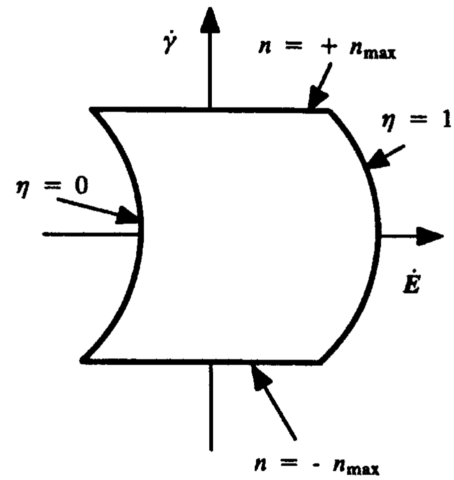

Fig. 2 Sample hodograph.

for $i = 0, \ldots, N-1$. Note that in the preceding equations and in the parameter set the controls are completely absent. The process of developing a representation of the form (14–16) by either explicit or implicit control elimination is discussed next.

Explicit Control Elimination

For given values of states at nodes, the load factor is first explicitly evaluated from Eq. (10) and then the throttle is explicitly evaluated from Eq. (9) with the value of load factor as obtained from Eq. (10) substituted into $\bar{D}_i$. Then the control constraints (11) and (12) are enforced, leading to the conditions (15) and (16). A sample hodograph space for the state rates that are directly controlled is shown in Fig. 2.

For this specific example, the number of parameters for the nonlinear programming problem using differential inclusions is $3(N+1)+1$ and the number of constraint equations is given by $3N+6$. For $N = 10$, the number of parameters is 34, and the number of constraints is 36.

Implicit Control Elimination

For the minimum time-to-climb example problem, the controls could be easily eliminated in an explicit fashion. For the most general problem, however, this is not always possible, and the control elimination may have to be performed in an iterative fashion. For the example of the F-15 minimum time-to-climb problem, the following steps need to be performed each time the optimizer requires an evaluation of the constraints.

1) For given states and state rates, the nonlinear 2×2 system (9) and (10) has to be solved for n_i and η_i.

2) Then the results have to be inserted into Eqs. (11) and (12) to evaluate the constraints.

The number of parameters and constraints for the nonlinear programming problem, as for the explicit control elimination method, remain at $3(N+1)+1$ and $3N+6$, respectively. The present method, however, adds the overhead of root-solving to eliminate the control variables.

Automation and generalization of the implicit control elimination is a current research topic. In the following paragraphs, the important issues and problems are briefly addressed.

The implicit control elimination scheme is most straightforward in cases where a one-to-one correspondence exists between the controls and the state rates. In this case, a zero-finding algorithm, as suggested earlier, can be used to solve the equations of motion $\dot{x} = f(x, u, t)$ for the controls u. The result can then be used to enforce the control constraints $g(x, u, t) = 0$ or $g(x, u, t) \leq 0$. In most cases such a one-to-one correspondence does not exist for the whole state vector, but only for a part of it. For example, let the system dynamics $\dot{x} = f(x, u, t)$ be decomposed to the form $\dot{y} = f_y(y, z, u, t)$ and $\dot{z} = f_z(y, z, t)$, such that a one-to-one correspondence exists between $\dot{y}$ and u (for arbitrary but fixed z and t). Then, in analogy, a zero-finding method can be applied to determine the controls u from $\dot{y} = f_y(y, z, u, t)$. The result can then be used to evaluate the control constraints.

The subset of the state rate space that can be influenced directly by the control variables may not always be the span of a subset of the components of the original state vector, as in the earlier example. Let the state and the control vectors be, say, n and m dimensional, respectively, with $m < n$. Let the state rates that can be influenced directly by the controls form an m-dimensional manifold in the n-dimensional state rate space. As long as a one-to-one correspondence exists between this manifold and the controls u, the controls may be eliminated uniquely through the minimization of $\|\dot{x} - f(x, u, t)\|_2$. The control constraints of the original optimal control problem may then be evaluated using the controls resulting from this minimization.

The most complicated case is encountered when the mapping from controls to state rates is not injective, i.e., when different control settings can yield the same state rate. First, note that in this case the optimal control problem is, in some sense, not well posed, as the optimal control is not determined uniquely. The optimal state and state rate histories, however, may still be determined uniquely. Also note that the transformation of such a problem into the differential inclusion format always eliminates the difficulties associated with nonuniqueness of the optimal control settings (as long as only the state histories are determined uniquely) simply because the controls are completely eliminated from the problem formulation. The remaining question is how this transformation can be achieved in a general, automated fashion. In theory, at least, the necessary steps can be stated as follows:

1) Find the controls such that the sum of the norms of the violations in the state equations and in the control constraints is minimized.

2) Use the controls obtained from step 1 to enforce the appropriate equality and inequality constraints, i.e., the state equations and the control constraints, respectively.

Note that this control elimination scheme requires smooth dependence of the residues (violations of individual constraints) on time t, state x, and state rate $\dot{x}$. It can be easily verified that the nonuniqueness of the controls that furnish the minimum residues does not cause any loss of smoothness in the optimization process.

Analytical Gradient Evaluation

Analytical evaluation of the gradient of the constraint with respect to the parameters is always recommended over an approximate numerical finite difference approach. The following paragraphs address this issue for the three approaches.

Let $\dot{x}_i = f(\bar{x}_i, u_i)$, where the $\bar{x}_i$ and $\dot{x}_i$ denote the state and the state rate vector at a nodal midpoint in the sense of Eq. (13), and let u_i denotes the control vector over the ith nodal interval. Let us assume a control constraint of the form $g(\bar{x}_i, u_i) \leq 0$, where the functions f and g are given. Let p denotes the parameter vector of the nonlinear programming problem. For the collocation method this parameter vector would contain the states x_i at discrete nodes and piecewise constant controls u_i along nodal intervals. For the differential inclusion approach the parameter vector would be made up of only the states x_i at discrete nodes.

Collocation Approach

For the collocation approach the analytical gradient $\partial g/\partial p$ is quickly provided by the chain rule of differentiation, i.e.,

$$\frac{\partial g}{\partial p} = \frac{\partial g}{\partial \bar{x}_i}\frac{\partial \bar{x}_i}{\partial p} + \frac{\partial g}{\partial u_i}\frac{\partial u_i}{\partial p} \tag{17}$$

The terms $\partial g/\partial \bar{x}_i$ and $\partial g/\partial u_i$ are obtained analytically by partial differentiation of the control constraint with respect to the states and controls, respectively. Since p is made up of the states x_i and controls u_i, the evaluation of $\partial \bar{x}_i/\partial p$ is simple and depends on the implicit integration rule. The evaluation of $\partial u_i/\partial p$ is trivial.

Explicit Control Elimination Approach

For the differential inclusion approach with explicit control elimination the analytical gradient is also obtained by Eq. (17). However, evaluation of $\partial u_i/\partial p$ is not as trivial as for the collocation scheme, since the controls at nodal intervals are not part of the parameter vector. By assumption of the explicit control elimination approach, the control u_i can be explicitly obtained from $\dot{x}_i = f(\bar{x}_i, u_i)$ as $u_i = h(\bar{x}_i, \dot{x}_i)$. This explicit relationship is used to evaluate $\partial u_i/\partial p$ as follows:

$$\frac{\partial u_i}{\partial p} = \frac{\partial h}{\partial \bar{x}_i}\frac{\partial \bar{x}_i}{\partial p} + \frac{\partial h}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial p} \tag{18}$$

The terms $\partial h/\partial \bar{x}_i$ and $\partial h/\partial \dot{x}_i$ are obtained analytically by partial differentiation of the explicit equation $u_i = h(\bar{x}_1, \dot{x}_i)$. Since p is made up of the states x_i at discrete nodes for this scheme, the partials $\partial \bar{x}_i/\partial p$ and $\partial \dot{x}_i/\partial p$ can be easily evaluated depending on the implicit integration rule used by the algorithm.

Implicit Control Elimination Approach

For the implicit control elimination approach the analytical gradient is also obtained from Eq. (17). However, evaluation of $\partial u_i/\partial p$ is not as trivial as for the collocation scheme, since the control at discrete nodes are not part of the parameter vector. By the implicit function theorem, we know that there exists a functional form $u_i = h(\bar{x}_i, \dot{x}_i)$ relating the control to the states and state rates. Let

$$F(\dot{x}_i, \bar{x}, u_i) = \dot{x}_i - f(\bar{x}_i, u_i) \equiv 0 \tag{19}$$

Then, partial differentiation of F with respect to p gives

$$\frac{\partial F}{\partial p} = 0 = \frac{\partial F}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial p} + \frac{\partial F}{\partial \bar{x}_i}\frac{\partial \bar{x}_i}{\partial p} + \frac{\partial F}{\partial u_i}\frac{\partial u_i}{\partial p} \tag{20}$$

from which $\partial u_i/\partial p$ can be evaluated as

$$\frac{\partial u_i}{\partial p} = -\left(\frac{\partial F}{\partial u_i}\right)^{-1}\left[\frac{\partial F}{\partial \bar{x}_i}\frac{\partial \bar{x}_i}{\partial p} + \frac{\partial F}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial p}\right] \tag{21}$$

Note that the existence of the inverse of the matrix $\partial F/\partial u_i$ is part of the assumptions required to apply the implicit function theorem.

Results and Discussion

All three methods, namely, the collocation based approach, differential inclusions with explicit control elimination, and differential inclusions with implicit control elimination, converge from trivial initial guesses (linearly interpolated states between boundary conditions). For the collocation method, the control variables had trivial guess values of zero at all nodal intervals. The differential inclusions approach does not require initial guesses for the controls. The number of nodes assumed for all the three cases was 11 ($N = 10$). Analytical gradients were not provided for the constraints. Initial guess of the scaled final time was 0.1 in all cases. Optimal scaled final time is approximately 2.57. Figure 3 shows the 11-node optimal trajectory for the F-15. All three methods converged to the same solution.

The number of major-loop iterations required by NPSOL to converge was similar for all three approaches (~40 iterations from the trivial initial guesses). The run time for the three approaches on a SunSparc 1+ compatible computer are for the explicit method 55 s, for the implicit method 180 s and for the collocation method 360 s.

Note that the implicit method generalizes the concept of control elimination but is penalized for the Newton iterations performed during constraint evaluations. The collocation approach has more parameters and constraints, and this contributes to the increased run time. The analytical constraint gradients can be easily evaluated for the collocation scheme in comparison to the explicit and implicit control elimination scheme.

Results from a detailed study[14] of the vertical rocket ascent problem (Goddard problem[6]), where a singular arc is part of the optimal trajectory is depicted in Table 1 for the differential inclusion method and Table 2 for the OTIS type collocation method. The number of nodes used was 11. Optimal final time for this problem is close to 0.2. For the differential inclusion approach, initial guess of states at all nodes were the states at launch pad. From these results it can be concluded that the differential inclusion approach shows enhanced convergence robustness in comparison to the OTIS type collocation approach for this specific problem. This statement, however, cannot be extended to the general class of singular control problems. For example, both the collocation and the differential inclusion approach showed good convergence behavior[14] for the minimum time reorientation problems with singular subarcs.[15]

Table 1 Convergence robustness of differential inclusion approach (Goddard problem)

Initial guess for final time	Number of iterations required for convergence
0.001	38
0.01	38
0.1	22
0.2	24
0.4	81
0.5	55
0.6	45
0.7	88
0.8	no convergence

Table 2 Convergence robustness of OTIS type approach (Goddard problem)

Initial guess	Number of iterations required for convergence
Perfect states at all nodes and final time; all controls zero	71
States at all nodes constant and equal to initial states; perfect final time; all controls zero	no convergence
Perfect states at all nodes; all controls zero	
$t_f = 0.2$	106
$t_f = 0.1$	36
$t_f = 0.01$	118
$t_f = 0.001$	118
States at all nodes constant and equal to initial states, except for final states (optimal); all controls zero	
$t_f = 0.1$	no convergence
$t_f = 0.2$	no convergence
$t_f = 0.4$	no convergence

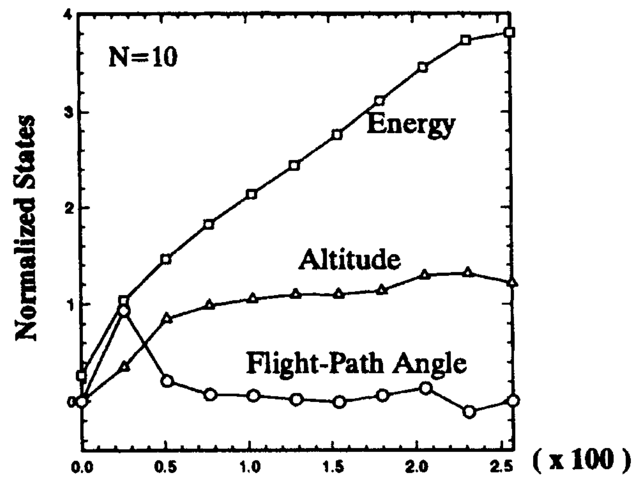

Fig. 3 Solution to the F-15 climb problem.

Generalization

For the F-15 climb-to-dash problem, the number of parameters and the number of constraints of the nonlinear programming problem resulting from the collocation-based approach are both larger than that for the differential inclusion approach. Can one generalize this statement to a larger class of optimal control problems?

Consider the following dynamic equations and control constraints:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + B(\boldsymbol{x})\boldsymbol{u} && \ldots, n \text{ equations} \\ \boldsymbol{u}_{\min} &\le \boldsymbol{u} \le \boldsymbol{u}_{\max} && \ldots, m \text{ equations} \end{aligned} \tag{22}$$

The controls map into state rates $\dot{\boldsymbol{x}}$ via an $n \times m$ matrix B, assumed to be of full rank. The function $\boldsymbol{f}(\cdot)$ is generally nonlinear. Without loss of generality, for the calculation of number of parameters and constraints of the associated discretized problem, $\boldsymbol{f}(\cdot)$ is assumed to be the zero vector. Assume an $N + 1$ nodal discretization for the problem. We consider three cases, i.e., a) $m > n$, b) $m = n$, and c) $m < n$, and examine the number of parameters and constraints arising for each case with and without control elimination. Any additional parameters required to define initial time, final time, or cost are ignored for this analysis.

Case a: ($m > n$)

The collocation approach has number of parameters N_p and number of constraints N_c given by

$$N_p = n * (N + 1) + m * N \tag{23}$$

and

$$N_c = n * N + m * N \tag{24}$$

For the explicit control elimination approach[16] with linearly appearing controls, we have

$$N_p = n * (N + 1) \tag{25}$$

and an upper bound on N_c given by

$$N_c = {}_mC_n * N \tag{26}$$

where C denotes the combinatorics operator, i.e., ${}_mC_n = m!/[n!(m-n)!]$. This approach reduces the number of parameters as compared to the collocation approach but introduces possibly a large number of constraints, especially if $m \gg n$.

Another approach of control elimination would be to augment the states $\boldsymbol{x}$ with $m - n$ dummy differential equations such that the new augmented matrix $\hat{B}$ is square and of full rank. Let $\boldsymbol{z}$ denote the augmented state vector. The new system of equations and constraints takes the following form:

$$\begin{aligned} \dot{\boldsymbol{z}} &= \hat{B}\boldsymbol{u} && \ldots, m \text{ equations} \\ \boldsymbol{u}_{\min} &\le \hat{B}^{-1}\dot{\boldsymbol{z}} \le \boldsymbol{u}_{\max} && \ldots, m \text{ equations} \end{aligned} \tag{27}$$

The number of parameters N_p and number of constraints N_c are given by

$$N_p = m * (N + 1) \tag{28}$$

and

$$N_c = m * N \tag{29}$$

It can be clearly seen that the number of parameters for large N is smaller in Eq. (28) than in Eq. (23). Moreover, the number of constraints given by Eq. (29) is always smaller than the number given by Eq. (24).

Case b: ($m = n$)

The collocation approach has number of parameters N_p and number of constraints N_c given by

$$N_p = n * (N + 1) + n * N \tag{30}$$

and

$$N_c = n * N + n * N \tag{31}$$

The differential constraints and control constraints for the differential inclusion approach take the following form:

$$\begin{aligned} \dot{\boldsymbol{x}} &= B\boldsymbol{u} && \ldots, n \text{ equations} \\ \boldsymbol{u}_{\min} &\le B^{-1}\dot{\boldsymbol{x}} \le \boldsymbol{u}_{\max} && \ldots, n \text{ equations} \end{aligned} \tag{32}$$

The number of parameters N_p and the number of constraints N_c are given by

$$N_p = n * (N + 1) \tag{33}$$

and

$$N_c = n * N \tag{34}$$

It can be observed from Eqs. (30–34) that the number of parameters and the number of constraints of the resulting nonlinear programming problem are $n * N$ smaller for the differential inclusion approach compared to the collocation approach.

Case c: ($m < n$)

The collocation approach has number of parameters N_p and number of constraints N_c given by

$$N_p = n * (N + 1) + m * N \tag{35}$$

and

$$N_c = n * N + m * N \tag{36}$$

For the differential inclusion approach, one strategy would be to augment the control vector $\boldsymbol{u}$ with $n - m$ dummy controls such that the new augmented matrix $\hat{B}$ is square and has full rank. The dummy controls are then constrained to be zero. Let $\hat{\boldsymbol{u}}$ denote the augmented control vector. The new system of equations and constraints takes the following form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \hat{B}\hat{\boldsymbol{u}} && \ldots, n \text{ equations} \\ \hat{\boldsymbol{u}}_{\min} &\le \hat{B}^{-1}\dot{\boldsymbol{x}} \le \hat{\boldsymbol{u}}_{\max} && \ldots, n \text{ equations} \end{aligned} \tag{37}$$

The bounds $\hat{\boldsymbol{u}}_{\min}$ and $\hat{\boldsymbol{u}}_{\max}$ have the first m components the same as $\boldsymbol{u}_{\min}$ and $\boldsymbol{u}_{\max}$, and the rest of the components are set to zero. The number of parameters N_p and number of constraints N_c are given by

$$N_p = n * (N + 1) \tag{38}$$

and

$$N_c = n * N \tag{39}$$

It can be inferred from Eqs. (35–39) that the number of parameters and the number of constraints of the resulting nonlinear programming problem are $m * N$ smaller for the differential inclusion approach compared to the collocation approach.

Conclusions

A minimum time-to-climb optimal control problem for an F-15 aircraft was used to compare and contrast direct solution methodologies via collocation and differential inclusions. In the former method, both states and controls are parameterized, whereas in the latter, only the states are parameterized. The controls are completely eliminated in the differential inclusion approach either explicitly or implicitly.

All three methods, namely, collocation, explicit control elimination, and implicit control elimination, converge from trivial initial guesses. The number of major-loop iterations required by the nonlinear programming software to converge was similar for all three approaches. The run time for the explicit control elimination method, however, was one-third that of the implicit method and one-sixth that of the collocation method. The implicit method generalizes the concept of control elimination but is penalized for the Newton iterations performed during constraint evaluations. The collocation approach has more parameters and constraints, and this contributes to its increased run time. A generalization to a specific class of optimal control problems with linearly appearing controls shows that the differential inclusion approach is, in general, better than the collocation approach in terms of the number of parameters and constraints involved in the nonlinear programming problem.

The conversion of an arbitrary optimal control problem into a differential inclusion format is nontrivial. Generalization and automation of the implicit control elimination methodology is a current research topic. Analytical constraint gradient evaluation is easier for the collocation method in comparison to the explicit and implicit control-elimination schemes.

To summarize, elimination of controls from the optimal control situation methodology has its merits in terms of robustness (especially for singular control problems), speed, and storage and has disadvantages in terms of ease of representation in the differential inclusion format and evaluation of analytical constraint gradients. Thus, an engineer/researcher should choose to keep or eliminate the controls based on the problem at hand.

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